

# Using Equivariant Obstruction Theory in Combinatorial Geometry

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**ABSTRACT.** A significant group of problems coming from the realm of Combinatorial Geometry can be approached fruitfully through the use of Algebraic Topology. From the first such application to Kneser's problem in 1978 by Lovász [18] through the solution of the Lovász conjecture [1], [9], many methods from Algebraic Topology have been developed. Specifically, it appears that the understanding of equivariant theories is of the most importance. The solution of many problems depends on the existence of an elegantly constructed equivariant map. For example, the following problems were approached by discussing the existence of appropriate equivariant maps. A variety of results from algebraic topology were applied in solving these problems. The methods used ranged from well known theorems like Borsuk-Ulam and Dold theorem to the integer / ideal-valued index theories. Recently equivariant obstruction theory has provided answers where the previous methods failed. For example, in papers [24] and [5] obstruction theory was used to prove the existence of different mass partitions. In this paper we extract the essence of the equivariant obstruction theory in order to obtain an effective *general position map* scheme for analyzing the problem of existence of equivariant maps. The fact that this scheme is useful is demonstrated in this paper with three applications:

(A) a "half-page" proof of the Lovász conjecture due to Babson and Kozlov [1] (one of two key ingredients is Carsten's map [9]),

(B) a generalization of the result of V. Makeev [19] about the sphere  $S^2$  measure partition by 3-planes (Section 2), and

(C) the new  $(a, b, a)$ , class of 3-fan 2-measures partitions (Section 3).

These three results, sorted by complexity, share the spirit of analyzing equivariant maps from spheres to complements of arrangements of subspaces.

## 1. Equivariant obstruction theory

The basic concept of any obstruction theory is to produce an invariant associated to a specific construction in such a way that the nature of the invariant determines whether the construction can or can not be performed. An (equivariant) obstruction theory considers two basic problems. For a finite group  $G$ , consider a relative  $G$ -cellular complex  $(X, A)$  such that the  $G$ -action on  $X \setminus A$  is **free**. Let  $Y$  be a  $G$ -space.

*Extension problem.* Let  $f : A \rightarrow Y$  be a  $G$ -map. Is there a  $G$ -map  $F : X \rightarrow Y$  such that  $f = F \circ i$ ? Here  $i : A \rightarrow X$  denotes the inclusion.

*Homotopy problem.* Let  $f_0 : X \rightarrow Y$  and  $f_1 : X \rightarrow Y$  be  $G$ -maps such that there is a  $G$ -homotopy  $h : I \times A \rightarrow Y$  from  $f_0|_A$  to  $f_1|_A$ . Is there a  $G$ -homotopy  $H : I \times X \rightarrow Y$  which extends  $h$ , i.e.,  $H|_{\{0\} \times X} = f_0$ ,  $H|_{\{1\} \times X} = f_1$  and  $H|_{I \times A} = h$ ?

The answer which obstruction theory provides is a sequence of obstruction elements living in equivariant cohomology. For the details about (equivariant) obstruction theory one can consult the expositions in [12, Section II. 3], [13, Chapter 7] and [22, Section V. 5].

**1.1. Equivariant homology and cohomology.** Let  $(X, A)$  be a relative  $G$ -cellular complex with a free action on  $X \setminus A$ . Let  $C_*(X, A)$  denote the integral cellular chain complex. The cellular free  $G$ -action on every skeleton of  $X \setminus A$  induces a free  $G$ -action on the chain complex  $C_*(X, A)$ . Therefore, the chain complex  $C_*(X, A)$  is actually a chain complex of free  $\mathbb{Z}[G]$ -modules.

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DEFINITION 1.1. Let  $(X, A)$  be a relative  $G$ -cellular complex such that the  $G$ -action on  $X \setminus A$  is free. Let  $M$  be a  $\mathbb{Z}[G]$ -module.

(A) The chain complex

$$(1.1) \quad \mathfrak{C}_*^G(X, A; M) = C_*(X, A) \otimes_{\mathbb{Z}[G]} M$$

is the equivariant chain complex of  $(X, A)$  with coefficients in  $M$ , and its homology  $\mathfrak{H}_*^G(X, A; M)$  is called the **equivariant homology** of  $(X, A)$  with coefficients in  $M$ .

(B) The cochain complex

$$(1.2) \quad \mathfrak{C}_G^*(X, A; M) = \text{Hom}_{\mathbb{Z}[G]}(C_*(X, A), M)$$

is the equivariant cochain complex of  $(X, A)$  with coefficients in  $M$ , and its homology  $\mathfrak{H}_G^*(X, A; M)$  is called the **equivariant cohomology** of  $(X, A)$  with coefficients in  $M$ .

A central example of a  $\mathbb{Z}[G]$ -module in our applications will come from a path connected,  $n$ -simple  $G$ -space  $Y$ . Being  **$n$ -simple** means that  $\pi_1(Y, y_0)$  acts trivially on  $\pi_n(Y, y_0)$  for every  $y_0 \in Y$ . The action of  $G$  on  $Y$  easily produces an action on the set of homotopy classes  $[S^n, Y]$ . Since  $Y$  is  $n$ -simple, the action can be extended to the homotopy group  $\pi_n Y \cong [S^n, Y]$ . Consequently there exists a  $\mathbb{Z}[G]$ -module structure on  $\pi_n Y$ .

The equivariant homology and cohomology groups of  $(X, A)$  can be interpreted as the ordinary homology and cohomology groups of the quotient pair  $(X/G, A/G)$  with appropriate local coefficients [12, p.112]. For  $M$  interpreted as a local coefficient system  $\tilde{M}$  on  $(X/G) \setminus (A/G)$ :

$$(1.3) \quad \mathfrak{H}_n^G(X, A; M) \cong H_n(X/G, A/G; \tilde{M}),$$

$$(1.4) \quad \mathfrak{H}_G^n(X, A; M) \cong H^n(X/G, A/G; \tilde{M}).$$

**1.2. The exact obstruction sequence.** Let  $n \geq 1$  be a fixed integer and  $Y$  a path-connected  $n$ -simple  $G$ -space. For every  $G$ -relative cell complex  $(X, A)$  with **free** action of  $G$  on  $X \setminus A$ , there exists an obstruction exact sequence

$$(1.5) \quad [X_{n+1}, Y]_G \longrightarrow \text{im}([X_n, Y]_G \rightarrow [X_{n-1}, Y]_G) \xrightarrow{\mathfrak{o}_G^{n+1}} \mathfrak{H}_G^{n+1}(X, A; \pi_n Y),$$

where  $X_k$  denotes the  $k$ -skeleton of  $X$ . The sequence is natural in  $X$  and  $Y$ . This exact sequence should be understood in the following way:

(A) Every  $G$ -map on the  $(n-1)$ -skeleton  $f : X_{n-1} \rightarrow Y$  which can be equivariantly extended to the  $n$ -skeleton  $f : X_n \rightarrow Y$  defines a unique element  $\mathfrak{o}_G^{n+1}(f)$  living in  $\mathfrak{H}_G^{n+1}(X, A; \pi_n Y)$ , called the **obstruction element**;

(B) The exactness of the sequence means that the obstruction element  $\mathfrak{o}_G^{n+1}(f)$  is zero if and only if there is a map in the homotopy class of the restriction  $f|_{X_{n-1}}$  which can be extended to the  $(n+1)$ -skeleton  $X_{n+1}$ .

The obstruction element can be introduced on the cochain level by a universal geometrical construction. Let  $[h] \in [X_n, Y]_G$ , let  $\varphi : (D^{n+1}, S^n) \rightarrow (X_{n+1}, X_n)$  be an attaching map and  $e \in C_{n+1}(X, A)$  the generator associated to the cell  $\varphi$ . Then the **obstruction cochain**  $\mathfrak{o}_G^{n+1}(h) \in \mathfrak{C}_G^{n+1}(X, A; \pi_n Y)$  of the map  $h$  is defined on  $e$  (with a little abuse of the notation) by

$$(1.6) \quad \mathfrak{o}_G^{n+1}(h)(e) = [h \circ \varphi] \in [S^n, Y].$$

It can be proved that the cohomology class of the obstruction cocycle is the obstruction element defined by the exact sequence (1.5).

**1.3. The primary obstruction.** Let  $n \geq 1$  be a fixed integer. The following proposition holds for an  $(n-1)$ -connected,  $n$ -simple  $G$ -space  $Y$ .

PROPOSITION 1.2. Let  $(X, A)$  be a relative  $G$ -cell complex with the action of  $G$  free on  $X \setminus A$ , and  $f : A \rightarrow Y$  any  $G$ -map.

(A) There exists a  $G$ -map  $h : X_n \rightarrow Y$  extending  $f$ , i.e.,  $h|_A = f$ .

(B) Every two  $G$ -extensions  $h$  and  $k : X_n \rightarrow Y$  of  $f$  are  $G$ -homotopic rel  $A$  on  $X_{n-1}$ , that is

$$\text{im}([X_n, Y]_G \rightarrow [X_{n-1}, Y]_G) = \{*\}.$$

(C) If  $H : I \times A \rightarrow Y$  is a  $G$ -homotopy between  $f : A \rightarrow Y$  and  $g : A \rightarrow Y$ , and  $h, k : X_n \rightarrow Y$  are the extensions of  $f$  and  $g$ , then there is a  $G$ -homotopy  $K : I \times X_{n-1} \rightarrow Y$  extending  $H$  between  $h|_{X_{n-1}}$  and  $k|_{X_{n-1}}$ .

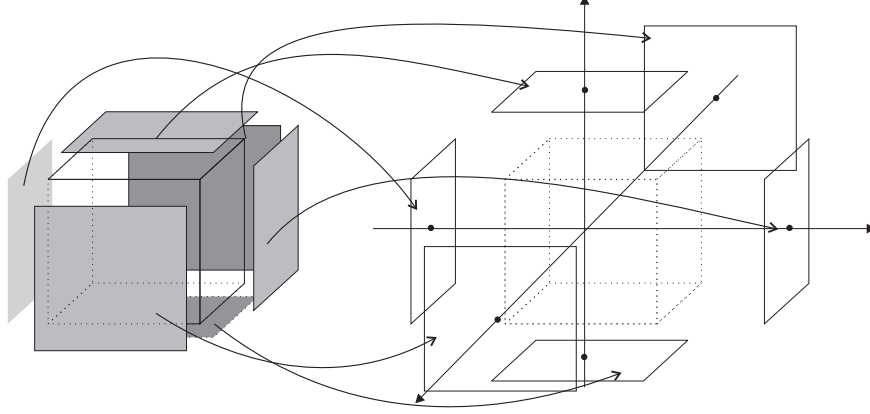


FIGURE 1. The obstruction cocycle for the map of a cube in the complement of the coordinate lines arrangement.

PROOF. (A) We extend  $f = f_0$  starting with the 0-skeleton and go up to the  $n$ -skeleton. The obstruction for lifting  $f_r : X_r \rightarrow Y$  from the  $r$ -skeleton to the  $(r+1)$ -skeleton lies in  $\mathfrak{C}_G^{r+1}(X, \pi_r Y)$ . Since  $Y$  is  $(n-1)$ -connected and  $n$ -simple,

$$\pi_r Y = 0 \text{ for all } 1 \leq r \leq n-1 \Rightarrow \mathfrak{C}_G^{r+1}(X, \pi_r Y) = 0 \text{ for all } 1 \leq r \leq n-1.$$

Hence, there is a  $G$ -map  $h : X_n \rightarrow Y$  extending  $f$ .

(B) Let  $h, k : X_n \rightarrow Y$  be  $G$ -extensions of  $f$ . A  $G$ -map

$$K : (\{0\} \times X_n) \cup (\{1\} \times X_n) \cup (I \times A) \rightarrow Y$$

can be defined by

$$\begin{aligned} K|_{\{0\} \times X_n} &= h, & K|_{\{1\} \times X_n} &= k, \\ K|_{\{t\} \times A} &= f, & \text{for every } t \in I. \end{aligned}$$

Now consider a relative  $G$ -cell complex  $(I \times X_n, (\{0\} \times X_n) \cup (\{1\} \times X_n) \cup (I \times A))$  and extend  $K$ . The assumptions on  $Y$  provide a  $G$ -map  $H$  from the  $n$ -skeleton  $(\{0\} \times X_n) \cup (\{1\} \times X_n) \cup (I \times A)$  of  $I \times X_n$  to  $Y$  which extends  $K$ . The restriction  $H|_{I \times X_{n-1}}$  is the required  $G$ -homotopy rel  $A$ .

(C) Considering a relative  $G$ -cell complex  $(I \times X_{n-1}, (\{0\} \times X_n) \cup (\{1\} \times X_n) \cup (I \times A))$  instead of  $(X, A)$  the statement becomes a direct consequence of the property (A).  $\square$

When

$$\text{im}([X_n, Y]_G \rightarrow [X_{n-1}, Y]_G) = \{*\},$$

the obstruction sequence (1.5) becomes

$$(1.7) \quad [X_{n+1}, Y]_G \longrightarrow \{*\} \xrightarrow{\mathfrak{o}_G^{n+1}} \mathfrak{H}_G^{n+1}(X, \pi_n Y).$$

The element  $\mathfrak{o}_G^{n+1}(*) \in \mathfrak{H}_G^{n+1}(X, \pi_n Y)$  is called the **primary obstruction** and does not depend on the map of the  $n$ -th skeleton.

COROLLARY 1.3. If  $\circ$  and  $*$  are  $G$ -actions on  $S^n$  and  $\circ$  is free, then there exists a  $G$ -map  $f : S^n \rightarrow S^n$  such that

$$f(g \circ x) = g * f(x)$$

for all  $g \in G$  and  $x \in S^h$ .

The Corollary is a direct consequence of the previous proposition statement (A).

**1.4. Equivariant Poincaré duality.** Let  $X$  be a compact  $n$ -dimensional free  $G$ -manifold. Then there is a version of the Poincaré duality isomorphism for the equivariant homology and cohomology of  $X$  with the coefficients in any  $G$ -module  $M$ .

THEOREM 1.4. Let  $X$  be a compact  $n$ -dimensional, simply connected, free  $G$ -manifold,  $\mathcal{Z}$ , a  $G$ -module,  $H_{n+1}(X, \mathbb{Z}) \cong \mathbb{Z}$ , and  $M$  a  $G$ -module. Then there is an isomorphism of the groups

$$\mathfrak{H}_G^k(X; M) \cong \mathfrak{H}_{n-k}^G(X; M \otimes \mathcal{Z})$$

for every  $k \in \{0, \dots, n\}$ .

PROOF. When  $M$  is interpreted as the local coefficient system  $\tilde{M}$ , the isomorphism (1.4) implies:

$$\mathfrak{H}_G^k(X; M) \cong H^k(X/G; \tilde{M}).$$

From [16, Theorem 4.18. p. 196] and [14, Proposition 1.40. p. 70] we know that  $X/G$  is a compact manifold with fundamental group  $\pi_1(X/G) \cong G$ . Since all the assumptions for applying Theorem 2.1 [21, p.23] are met,

$$H^k(X/G; \tilde{M}) \cong H_{n-k}^t(X/G; \tilde{M}).$$

Careful reading of the definition of the modified homology  $H_*^t(\cdot)$ , [21, p. 21] implies that

$$H_{n-k}^t(X/G; \tilde{M}) = H_{n-k}(X/G; \tilde{M} \otimes \mathcal{Z}).$$

The isomorphism (1.3) concludes the proof

$$H_{n-k}(X/G; \tilde{M} \otimes \mathcal{Z}) \cong \mathfrak{H}_{n-k}^G(X; M \otimes \mathcal{Z}).$$

□

**1.5. The existence of a  $G$ -map  $M \rightarrow W \setminus \Sigma$  from a manifold to a complement.** Let  $M$  be a connected,  $(n+1)$ -dimensional, compact free  $G$ -manifold,  $W$  a  $d$ -dimensional smooth  $G$ -manifold, and  $\Sigma$  the union of a finite  $G$ -invariant arrangement  $\mathcal{S} = \{S_i | i \in I\}$  of the  $(d-n-1)$ -dimensional smooth submanifolds. Let us also assume that

- (A) the complement  $W \setminus \Sigma$  is  $n$ -simple, paracompact space,
- (B) the complement  $W \setminus \Sigma$  is  $(n-1)$ -connected,
- (C) the tangent spaces of the submanifolds  $S_i$  in any mutual intersection point do not coincide, and
- (D)  $H_n(W, \mathbb{Z}) = 0$ .

The question we consider is whether there is a  $G$ -map  $M \rightarrow W \setminus \Sigma$ .

1. Since the complement  $W \setminus \Sigma$  is  $(n-1)$ -connected by assumption, the problem of the existence of a  $G$ -map  $M \rightarrow W \setminus \Sigma$  depends only on primary obstruction. The obstruction exact sequence, like in (1.7), has the form

$$[M, W \setminus \Sigma]_G \longrightarrow \{*\} \xrightarrow{\circ_G^{n+1}} \mathfrak{H}_G^{n+1}(M, \pi_n(W \setminus \Sigma)).$$

The assumptions (A) and (B) on the space  $W \setminus \Sigma$  and the Hurewicz theorem imply that  $\pi_n(W \setminus \Sigma) \cong H_n(W \setminus \Sigma, \mathbb{Z})$  as  $G$ -modules. Thus, the obstruction element  $\circ_G^{n+1}(*)$  lives in the group  $\mathfrak{H}_G^{n+1}(M, H_n(W \setminus \Sigma, \mathbb{Z}))$ , where the natural  $G$ -structure on  $H_n(W \setminus \Sigma, \mathbb{Z})$  is assumed.

Consider the equivariant cohomology group  $\mathfrak{H}_G^{n+1}(M, H_n(W \setminus \Sigma, \mathbb{Z}))$ . Since  $M$  is a free  $G$ -manifold by assumption, Theorem 1.4 provides an isomorphism

$$\mathfrak{H}_G^{n+1}(M, H_n(W \setminus \Sigma, \mathbb{Z})) \cong \mathfrak{H}_0^G(M, H_n(W \setminus \Sigma, \mathbb{Z}) \otimes \mathcal{Z})$$

where  $\mathcal{Z}$  is the  $G$ -module  $H_{n+1}(M, \mathbb{Z}) \cong \mathbb{Z}$ . The isomorphism (1.3) implies that

$$\mathfrak{H}_0^G(M, H_n(W \setminus \Sigma, \mathbb{Z}) \otimes \mathcal{Z}) \cong H_0(M/G; H_n(W \setminus \Sigma, \mathbb{Z}) \otimes \mathcal{Z}).$$

Since  $G \cong \pi_1(M/G)$  an application of Proposition 5.14. of [13, p.107] (or alternatively [7, Exercise 1, p.44] and [7, (1.5), p.57]) provides an isomorphism

$$(1.8) \quad H_0(M/G; H_n(W \setminus \Sigma, \mathbb{Z}) \otimes \mathcal{Z}) \cong (H_n(W \setminus \Sigma, \mathbb{Z}) \otimes \mathcal{Z})_G.$$

Thus the obstruction element lives in a group of coinvariants of the first non-trivial reduced homology group of the target space  $W \setminus \Sigma$ ,

$$\circ_G^{n+1}(*) \in \mathfrak{H}_G^{n+1}(M, H_n(W \setminus \Sigma, \mathbb{Z})) \cong (H_n(W \setminus \Sigma, \mathbb{Z}) \otimes \mathcal{Z})_G$$

2. The situation where the primary obstruction is the only obstruction, as in our case, has the advantage of not depending on the particular  $G$ -map on the  $n$ -th skeleton.

DEFINITION 1.5. Let a  $G$ -map  $f : M \rightarrow W$  satisfy the following conditions

- (A)  $f(M_n) \subset W \setminus \Sigma$ ,
- (B)  $f(M) \cap \Sigma$  is a finite set of points,
- (C)  $(\forall x \in f(M) \cap \Sigma)(\forall S \in \mathcal{S}) \{x\} = S \cap f(M) \Rightarrow$  intersection is transversal at  $x$ ,
- (D)  $(\forall x \in f(M) \cap \Sigma)(\forall S_1, S_2 \in \mathcal{S}) \{x\} = S_1 \cap S_2 \cap f(M) \Rightarrow \text{codim}_{S_1}(S_1 \cap S_2) = 1$ .

We then say that  $f : M \rightarrow W$  is a map in **general position** with respect to  $\Sigma$ .

Condition (D) allows the intersection points  $f(M) \cap \Sigma$  to belong to the lower strata of the arrangement  $\mathcal{S}$ . This forces the introduction of broken point classes along point classes as possible results of evaluation of the obstruction cocycle.

**3.** The notion of point and broken point classes was introduced in [5] and [4] for the complements of arrangements of linear spaces. We extend this definition to the present setting. Consider  $x \in \Sigma$ . There are elements  $S_1, \dots, S_k$  in  $\mathcal{S}$  such that  $x \in S_1 \cap \dots \cap S_k$  and  $\text{codim}_{S_i}(S_1 \cap \dots \cap S_k) = 1$ . Let  $D_1, \dots, D_k$  denote disks in fibers at the point  $x$  of tubular neighborhoods of the submanifolds  $S_1, \dots, S_k$  such that for all  $i, j \in \{1, \dots, k\}$  we have  $S_i \cap D_j = \{x\}$ . The smoothness assumptions on  $W$  and elements of the arrangement  $\mathcal{S}$  guarantee the existence of the above construction [6, Theorem 11.14. p.100]. The fundamental class of the pair  $(D_i, \partial D_i)$  determines a homology class in  $H_{n+1}(W, W \setminus \Sigma; \mathbb{Z})$  that we denote by  $[x, D_i]$  and call the **point class** of  $x$  determined by  $D_i$ . The homotopy axiom implies that  $[x, D_i]$  does not change if  $x$  is moved inside the connected component of  $S_i \setminus \bigcup \{S \neq S_i \mid S \in \mathcal{S}\}$ . Because of the assumption (D), at the beginning of the section, that  $H_n(W, \mathbb{Z}) = 0$ , the epimorphism  $H_{n+1}(W, W \setminus \Sigma; \mathbb{Z}) \rightarrow H_n(W \setminus \Sigma)$  determines the class  $\|x, D_i\| := \partial[x, D_i]$  which is also called the **point class** of  $x$  determined by  $D_i$ . Thus all point classes  $\|x, D_i\|$  are born as  $[x, D_i]$  classes, but  $\|x, D_i\|$  can be zero while  $[x, D_i] \neq 0$ .

**PROPOSITION 1.6.** Consider a  $G$ -map  $f : M \rightarrow W$  in general position with respect to  $\Sigma$ . The obstruction element  $\mathfrak{o}_G^{n+1}(f|_{M^n})$  is the equivariant Poincaré dual of the point set  $f^{-1}(f(M) \cap \Sigma)$  understood as a chain in the group

$$\mathfrak{H}_0^G(M, H_n(W \setminus \Sigma, \mathbb{Z}) \otimes \mathcal{Z})$$

with the appropriate coefficients in the group of coefficients.

**4.** To compute the obstruction cocycle and the obstruction element, we have to choose at least one equivariant cell structure on  $M$  compatible with the given action. We choose two equivariant cell structures connected by a cellular map. To simplify the exposition, from now on we assume that  $\mathcal{Z}$  is a trivial  $G$ -module.

Usually, the first equivariant cell structure induced on  $M$  is a simplicial one. It is used to define a piecewise affine  $G$ -map in general position  $f : M \rightarrow W$ . The advantage of the simplicial structure is that the map is completely determined by the images of the vertex orbits and the requirement that the map is piecewise affine.

The second equivariant cell structure should satisfy the requirement that the top dimensional group of chains is generated equivariantly by a single cell  $e$ , and therefore

$$C_{n+1}^G(M, \mathbb{Z}) = \mathbb{Z}[G]e.$$

Such a structure will be called an **economic**  $G$ -structure of  $M$ . The economic  $G$ -cell structure does not have to exist.

**5.** The obstruction cocycle  $\mathfrak{o}_G^{n+1}(f)$  is computed using the simplicial cell structure and the geometric definition of the obstruction cocycle.

**PROPOSITION 1.7.** Let  $f : M \rightarrow W$  be a map in general position in respect to  $\Sigma$ . Then for a  $(n+1)$ -simplex  $\sigma$  of  $M$  the following formula holds

$$(1.9) \quad \mathfrak{o}_G^{n+1}(f)(\sigma) = \sum_{x \in f^{-1}(f(\sigma) \cap \Sigma)} \mathbf{I}(e, S_{f(x)}) \|f(x), f(\sigma)\| \in H_n(W \setminus \Sigma, \mathbb{Z}).$$

Here  $\mathbf{I}(e, S_{f(x)})$  denotes the intersection number of the image  $f(e)$  and the appropriate oriented element  $S_{f(x)}$  of the arrangement  $\mathcal{S}$ .

A cellular map between two cell structures allows evaluation of the obstruction cocycle in both structures. In an economic  $G$ -structure, every maximal cochain is determined by its value on the equivariant generator  $e$ . Therefore the obstruction cocycle can be treated as an element of the coefficient group  $\mathfrak{o}_G^{n+1}(f)(e) \in H_n(W \setminus \Sigma, \mathbb{Z})$  expressed as a linear combination of the point classes. The obstruction element is the class  $[\mathfrak{o}_G^{n+1}(f)(e)]$ , along the quotient homomorphism  $H_n(W \setminus \Sigma, \mathbb{Z}) \rightarrow H_n(W \setminus \Sigma, \mathbb{Z})_G$ .

**THEOREM 1.8.** *There exists a  $G$ -map  $M \rightarrow W \setminus \Sigma$  if and only if  $[\mathfrak{o}_G^{n+1}(f)(e)] \in H_n(W \setminus \Sigma, \mathbb{Z})_G$  is zero.*

**1.6. Example: Lovász conjecture.** The first proof of the Lovász conjecture was given by Babson and Kozlov [1]. A revealingly simple proof was given by Schultz, [9], [10]. Kozlov gave another proof in [17].

**THEOREM 1.9.** *Let  $\Gamma$  be a graph,  $r \geq 1$  and  $n \geq 2$ . If  $\text{Hom}(C_{2r+1}, \Gamma)$  is  $(n-1)$ -connected, then  $\Gamma$  is not  $(n+2)$ -colorable.*

Here  $C_{2r+1}$  is a circular graph with  $2r+1$  edges. Let us assume that  $\Gamma$  is  $(n+2)$ -colorable. This means that there exists a graph homomorphism  $\Gamma \rightarrow K_{n+2}$ , and consequently a map  $\text{Hom}(H, \Gamma) \rightarrow \text{Hom}(H, K_{n+2})$

for every graph  $H$ . When we put  $C_{2r+1}$  with a  $\mathbb{Z}_2$ -action instead of  $H$ , we end up with the  $\mathbb{Z}_2$ -equivariant map

$$(1.10) \quad \text{Hom}(C_{2r+1}, \Gamma) \rightarrow \text{Hom}(C_{2r+1}, K_{n+2}).$$

The assumption that  $\text{Hom}(C_{2r+1}, \Gamma)$  is  $(n-1)$ -connected implies the existence of a  $\mathbb{Z}_2$  map

$$(1.11) \quad S^n \rightarrow \text{Hom}(C_{2r+1}, \Gamma).$$

where  $S^n$  is equipped with the antipodal action. Thus to prove the Lovász conjecture it is enough to prove that there is no  $\mathbb{Z}_2$ -map

$$(1.12) \quad S^n \rightarrow \text{Hom}(C_{2r+1}, K_{n+2}).$$

In [9], C. Schultz proved the nonexistence of the map (1.12) by comparing the complex  $\text{Hom}(C_{2r+1}, K_{n+2})$  with the complement of the torus arrangement and then performing some characteristic class computations. Let us reproduce this beautiful construction and substitute characteristic class computations with obstruction theory. Let  $X_{r,n} = (S^n)^r$  be a torus, and  $A_{r,n}$  the union of the arrangement of the following  $r$  subtoruses

$$A_{r,n}^i = \{x \in X_{r,n} \mid x_i = -x_{i+1}\}, 0 \leq i \leq r-2 \text{ and } A_{r,n}^{r-1} = \{x \in X_{r,n} \mid x_{r-1} = x_0\}.$$

If we define a  $\mathbb{Z}_2 = \langle \omega \rangle$  action on  $X_{r,n}$  by

$$\omega \cdot (x_0, x_1, x_2, \dots, x_{r-1}) = (-x_0, x_{r-1}, x_{r-2}, \dots, x_1)$$

it is apparent that  $A_{r,n}$  is a  $\mathbb{Z}_2$ -invariant subspace of  $X_{r,n}$ . Proposition 2.9 of [9] says that there exists a  $\mathbb{Z}_2$ -map

$$(1.13) \quad \text{Hom}(C_{2r+1}, K_{n+2}) \rightarrow X_{r,n} \setminus A_{r,n}.$$

Thus, the assumption that  $\Gamma$  is  $(n+2)$ -colorable implies the existence of a  $\mathbb{Z}_2$ -equivariant map

$$S^n \rightarrow X_{r,n} \setminus A_{r,n}.$$

The following theorem provides a contradiction which implies Lovász conjecture.

**THEOREM 1.10.** *There is no  $\mathbb{Z}_2$ -equivariant map  $S^n \rightarrow X_{r,n} \setminus A_{r,n}$ .*

**PROOF.** The codimension of  $A_{r,n}$  inside  $X_{r,n} = (S^n)^r$  is  $n$ . Since  $A_{r,n}$  is compact, locally contractible space and  $H^i(A_{r,n}, \mathbb{Z}) = 0$  for  $i > r(n-1)$  then Poincaré - Lefschetz Duality, ([6, Corollary 8.4. p.352], [14, Proposition 3.46. p.254]), combined with long exact sequence in cohomology, implies that the complement  $X_{r,n} \setminus A_{r,n}$  is  $n-2$  connected. Thus, the existence of a  $\mathbb{Z}_2$ -map  $S^n \rightarrow X_{r,n} \setminus A_{r,n}$  is determined by the primary obstruction. Let  $\xi = (0, \dots, 0, 1) \in S^n$ . A map  $f : S^n \rightarrow X_{r,n}$  defined by

$$f(x) = (x, \xi, \dots, \xi)$$

is a  $\mathbb{Z}_2$ -map in general position and does not hit lower strata of the arrangement. Indeed, the intersection contains two points

$$f(S^n) \cap A_{r,n} = \{(\xi, \xi, \dots, \xi), (-\xi, \xi, \dots, \xi)\}.$$

Moreover,

$$(\xi, \xi, \dots, \xi) \in A_{r,n}^{r-1} \setminus \bigcup_{i \neq r-1} A_{r,n}^i, \quad (-\xi, \xi, \dots, \xi) \in A_{r,n}^0 \setminus \bigcup_{i \neq 0} A_{r,n}^i$$

and  $\omega \cdot (\xi, \dots, \xi) = (-\xi, \xi, \dots, \xi)$ . Proposition 1.6 implies that the obstruction element living in

$$\mathfrak{H}_{\mathbb{Z}_2}^n(S^n, H_{n-1}(X_{r,n} \setminus A_{r,n}, \mathbb{Z}))$$

is the equivariant Poincaré dual of the (orbit of the) point in

$$\mathfrak{H}_0^{\mathbb{Z}_2}(S^n, H_{n-1}(X_{r,n} \setminus A_{r,n}, \mathbb{Z}) \otimes \mathcal{Z}).$$

The isomorphisms [7, Exercise 1, p.44], [7, (1.5), p.57], and the fact that

$$H_{n-1}(X_{r,n} \setminus A_{r,n}, \mathbb{Z}) \neq 0$$

imply

$$\begin{aligned} \mathfrak{H}_0^{\mathbb{Z}_2}(S^n, H_{n-1}(X_{r,n} \setminus A_{r,n}, \mathbb{Z}) \otimes \mathcal{Z}) &\cong H_0(\mathbb{Z}_2, H_{n-1}(X_{r,n} \setminus A_{r,n}, \mathbb{Z}) \otimes \mathcal{Z}) \\ &\cong (H_{n-1}(X_{r,n} \setminus A_{r,n}, \mathbb{Z}) \otimes \mathcal{Z})_{\mathbb{Z}_2} \\ &\neq 0. \end{aligned}$$

Since the orbit of a point is a generator of the 0-homology, the obstruction element is not zero and the map does not exist.  $\square$

REMARK 1.11. In this example we were fortunate, because the intersection of the image of the general position map and the forbidden set was just an orbit of a point. That allowed a direct application of Proposition 1.6.

## 2. Partition of Sphere measures by Hyperplanes

This chapter contains an extension of Makeev's result [19].

A **proper** Borel measure on the sphere  $S^2$  is a Borel measure  $\mu$  such that

- (A)  $\mu([a, b]) = 0$  for any circular arc  $[a, b] \subset S^2$ , and
- (B)  $\mu(U) > 0$  for each nonempty open set  $U \subset S^2$ .

Let  $\Pi_1, \Pi_2$  and  $\Pi_3$  be three planes through the origin in  $\mathbb{R}^3$ . Planes are in a **fan position** if they intersect along a common line. The planes in a fan position cut the sphere  $S^2$  in six parts  $\sigma_1, \dots, \sigma_6$  naturally oriented up to a cyclic permutation.

PROBLEM 2.1. Find all the six-tuples  $\alpha = (\alpha_1, \dots, \alpha_6) \in \mathbb{N}^6$  such that  $\frac{\alpha_1}{\alpha_1 + \dots + \alpha_6} + \dots + \frac{\alpha_6}{\alpha_1 + \dots + \alpha_6} = 1$ , and that for any proper Borel probability measure  $\mu$  on the sphere  $S^2$  there exist three planes in a fan position, with angular sectors having the prescribed amount of the measure, i.e., for all  $i \in \{1, \dots, 6\}$ ,

$$\mu(\sigma_i) = \frac{\alpha_i}{\alpha_1 + \dots + \alpha_6}.$$

The six-tuples which satisfy these conditions are **solutions** of the problem.

The existence of the equipartition solution was proved by V. V. Makeev [19]. Modifying the configuration space and the test map, we prove in section 2.2 that beside  $(1, 1, 1, 1, 1, 1)$  there exist at least two more solutions.

THEOREM 2.2. *Let  $\mu$  be a proper Borel probability measure on the sphere  $S^2$ . Then there are three planes intersecting along a line such that the ratio of the measure  $\mu$  in the angular sectors cut by the planes is*

$$(A) (1, 1, 2, 1, 1, 2) \quad (B) (1, 2, 2, 1, 2, 2).$$

**2.1. The configuration space / test map scheme.** We use the configuration space / test map scheme to reduce the partition problem to an equivariant one. The basic idea, which we modify, comes from the papers of Imre Bárány and Jiří Matoušek [2], [3].

The  **$k$ -fan** ( $l; \Pi_1, \Pi_2, \dots, \Pi_k$ ) in  $\mathbb{R}^3$  (or on  $S^2$ ) is formed from an oriented line through the origin  $l$  and  $k$  closed half planes  $\Pi_1, \Pi_2, \dots, \Pi_k$  which intersect along the common boundary  $l = \partial\Pi_1 = \dots = \partial\Pi_k$ . The intersection of a  $k$ -fan with the sphere  $S^2$  is also equally called a  $k$ -fan. Thus, the collection  $(x; l_1, \dots, l_k)$  of a point  $x \in S^2$  and  $k$  great semicircles  $l_1, \dots, l_k$  emanating from  $x$  is also a  $k$ -fan. Sometimes instead of great semicircles we use:

- (A) open angular sectors  $\sigma_i$  between  $l_i$  and  $l_{i+1}$ ,  $i = 1, \dots, k$ ; or
- (B) tangent vectors  $t_i \in T_x S^2$  which are determined by the great semicircle curves  $l_i$ ,  $i = 1, \dots, k$ .

Here  $T_x S^2$  denotes the tangent space at a point  $x \in S^2$ . So there are four equally useful notations for a single  $k$ -fan, and we prefer the tangent vector notation  $(x; t_1, \dots, t_k)$ . The space of all  $k$ -fans in  $\mathbb{R}^3$  will be denoted by  $F_k$ .

**The configuration space.** For a proper Borel probability measure  $\mu$  on  $S^2$  and  $n > 1$ , the configuration space is defined by

$$X_{\mu, n} = \{(x; t_1, \dots, t_n) \in F_n \mid (\forall i = 1, \dots, n) \mu(\sigma_i) = \frac{1}{n}\}.$$

Since every  $n$ -fan  $(x; t_1, \dots, t_n)$  of the configuration space  $X_{\mu, n}$  is completely determined by the pair  $(x, t_1) \in S^2 \times T_x S^2$  and the measure  $\mu$ , there is a homeomorphism  $X_{\mu, n} \cong V_2(\mathbb{R}^3)$ .

**The test map.** Let us fix the "symmetric" six-tuple  $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^6$  such that  $\alpha_1 + \alpha_2 + \alpha_3 = \frac{n}{2}$ . The test map for our problem is defined by

$$\Phi : X_{\mu, n} \rightarrow W_n = \{x \in \mathbb{R}^n \mid \sum_i x_i = 0\}, \quad \Phi(x; t_1, \dots, t_n) = (\theta_1 - \frac{2\pi}{n}, \dots, \theta_n - \frac{2\pi}{n}),$$

where  $\theta_i$  is the angle between tangent vectors  $t_i$  and  $t_{i+1}$  in the tangent plane  $T_x S^2$ . Here we assume that  $t_{n+1} = t_1$ .

**The action.** The dihedral group  $\mathbb{D}_{2n} = \langle j, \varepsilon \mid \varepsilon^n = j^2 = 1, \varepsilon j = j \varepsilon^{n-1} \rangle$  acts both on the configuration space  $X_{\mu, n}$  and on the hyperplane  $W_n$  in the following way

$$\begin{cases} \varepsilon(x; t_1, \dots, t_n) = (x; t_n, t_1, \dots, t_{n-1}) \\ j(x; t_1, \dots, t_n) = (-x; t_1, t_n, \dots, t_2) \end{cases} \quad , \quad \begin{cases} \varepsilon(x_1, \dots, x_n) = (x_n, x_1, \dots, x_{n-1}) \\ j(x_1, \dots, x_n) = (x_n, \dots, x_2, x_1) \end{cases} ,$$

for  $(x; t_1, \dots, t_n) \in X_{\mu, n}$  and  $(x_1, \dots, x_n) \in W_n$ . The action of  $\mathbb{D}_{2n}$  on  $X_{\mu, n}$  is **free** and the test map  $\Phi$  is equivariant.

**The test space.** The test space in this symmetric problem is the union  $\bigcup \mathcal{A} \subset W_n$  of the smallest  $\mathbb{D}_{2n}$ -invariant arrangement  $\mathcal{A}$ , which contains a linear subspace  $L \subset W_n$  defined by the equalities

$$x_1 + \dots + x_{\frac{n}{2}} = x_{\alpha_1+1} + \dots + x_{\alpha_1+\frac{n}{2}} = x_{\alpha_1+\alpha_2+1} + \dots + x_{\alpha_1+\alpha_2+\frac{n}{2}} = 0.$$

We have proved the basic proposition of the configuration space / test map scheme.

**PROPOSITION 2.3.** Let  $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^6$  be a symmetric 6-tuple such that  $\alpha_1 + \alpha_2 + \alpha_3 = \frac{n}{2}$ . If there is no  $\mathbb{D}_{2n}$ -equivariant map

$$V_2(\mathbb{R}^3) \rightarrow W_n \setminus \bigcup \mathcal{A},$$

then for every proper Borel probability measure on the sphere  $S^2$  there exist three planes in a fan position with angular sectors such that

$$(\forall i \in \{1, \dots, 6\}) \mu(\sigma_i) = \frac{\alpha_i}{n}.$$

The extension of scalars from homological algebra (as shown in [5] and [4]) allows us to prove the following equivalence. Let  $\mathbb{Q}_{4n}$  denote the generalized quaternion group  $\langle \epsilon, j \rangle \subset S^3$ , (section 4.1).

**PROPOSITION 2.4.** Following maps jointly exist or do not exist:

$$\text{a } \mathbb{D}_{2n}\text{-map } V_2(\mathbb{R}^3) \rightarrow W_n \setminus \bigcup \mathcal{A} \text{ and a } \mathbb{Q}_{4n}\text{-map } S^3 \rightarrow W_n \setminus \bigcup \mathcal{A}.$$

The group  $\mathbb{Q}_{4n}$  acts on  $S^3$  as a subgroup and on  $W_n$  via the quotient homomorphism  $\mathbb{Q}_{4n} \rightarrow \mathbb{Q}_{4n}/H \cong \mathbb{D}_{2n}$  (see Appendix).

**2.2. Proof of Theorem 2.2.** According to Propositions 2.3 and 2.4 it is enough to prove that there is no  $\mathbb{Q}_{4n}$ -map  $S^3 \rightarrow W_n \setminus \bigcup \mathcal{A}$ . Here  $\mathcal{A}$  is the minimal  $\mathbb{Q}_{4n}$  ( $=\mathbb{D}_{2n}$ ) arrangement containing the subspace  $L$  defined

(A) for  $n = 8$  and  $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2, \alpha_3) = (1, 1, 2, 1, 1, 2)$ , by the equations:

$$x_1 + x_2 + x_3 + x_4 = x_2 + x_3 + x_4 + x_5 = x_3 + x_4 + x_5 + x_6 = 0;$$

(B) for  $n = 10$  and  $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2, \alpha_3) = (1, 2, 2, 1, 2, 2)$ , by the equations:

$$x_1 + \dots + x_5 = x_2 + \dots + x_6 = x_4 + \dots + x_8 = 0.$$

The codimension of  $\mathcal{A}$  inside  $W_n$  in both cases is 3, so the complement  $W_n \setminus \bigcup \mathcal{A}$  is 1-connected (by Poincaré - Lefschetz Duality [6, Corollary 8.4, p.352], [14, Proposition 3.46, p.254] or the Goresky-MacPherson formula). Therefore the primary obstruction is responsible for the existence of  $\mathbb{Q}_{4n}$ -maps  $S^3 \rightarrow W_n \setminus \bigcup \mathcal{A}$ , and the obstruction exact sequence has the form

$$[S^3, W_n \setminus \bigcup \mathcal{A}]_{\mathbb{Q}_{4n}} \longrightarrow \{*\} \xrightarrow{\circ_{\mathbb{Q}_{4n}}^3} \mathfrak{H}_{\mathbb{Q}_{4n}}^3(S^3, H_2(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z})).$$

Since  $\mathbb{Q}_{4n} \subset S^3$  acts on  $S^3$  as a subgroup and  $S^3$  is connected, the  $\mathbb{Q}_{4n}$ -module  $\mathcal{Z}$  is trivial. The Equivariant Poincaré duality isomorphism (Theorem 1.4) implies

$$\begin{aligned} \mathfrak{H}_{\mathbb{Q}_{4n}}^3(S^3, H_2(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z})) &\cong \mathfrak{H}_0^{\mathbb{Q}_{4n}}(S^3, H_2(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z}) \otimes \mathcal{Z}) \\ &\cong H_2(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z})_{\mathbb{Q}_{4n}}. \end{aligned}$$

Thus, in both cases we are going to define a map in general position, compute the obstruction cocycle and identify its obstruction element inside a group of coinvariants. All computations are done by the Mathematica 5.0 package.

**2.2.1. Case  $n = 8$  and  $\alpha = (1, 1, 2, 1, 1, 2)$ .** Let us define a map  $f : S^3 \rightarrow W_8$  on the vertex  $t (= a_1$  in Appendix) by

$$f(t) = (-3, 3, -1, 1, 1, -2, 2, -1)$$

and extend it equivariantly. For example  $f(jt) = (-1, 2, -2, 1, 1, -1, 3, -3)$ .

For the subspace  $L$  defined by

$$x_1 + x_2 + x_3 + x_4 = x_2 + x_3 + x_4 + x_5 = x_3 + x_4 + x_5 + x_6 = \sum_{i=1}^8 x_i = 0,$$

the arrangement  $\mathcal{A}$  is the minimal  $\mathbb{Q}_{32}$ -arrangement containing  $L$ . It has four maximal elements  $L$ ,  $\epsilon L$ ,  $\epsilon^2 L$  and  $\epsilon^3 L$ . This can easily be seen from the set equalities  $\epsilon^4 L = L$  and  $\epsilon^2 L = jL$ . The intersection  $L \cap \epsilon L \cap \epsilon^2 L \cap \epsilon^3 L$  is a codimension 1 subspace of  $L$ ,  $\epsilon L$ ,  $\epsilon^2 L$ ,  $\epsilon^3 L$ . Thus, the Hasse diagram of the intersection poset of the arrangement  $\mathcal{A}$  with the reversed order is as in Figure 2.

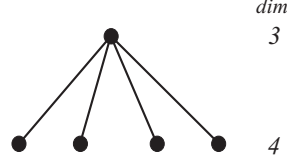


FIGURE 2.

Now we intersect the image under  $f$  of the maximal cell  $e = ([t, \epsilon t] \cup \dots \cup [\epsilon^7 t, \epsilon^8 t]) * [jt, \epsilon jt]$  with the test space  $\bigcup \mathcal{A} = L \cup \epsilon L \cup \epsilon^2 L \cup \epsilon^3 L$ . The results of the  $8 \times 4 = 32$  intersections  $f([\epsilon^i t, \epsilon^{i+1} t] * [jt, \epsilon jt]) \cap \epsilon^r L$  are summarized in the following table:

	$f([\epsilon^i t, \epsilon^{i+1} t] * [jt, \epsilon jt]) \cap \epsilon^r L$		
$\mathbf{i/r}$	preimage of the intersection point	intersection point in $W_8$	
1/2	$\frac{3}{7}jt + \frac{1}{14}\epsilon jt + \frac{1}{14}\epsilon t + \frac{3}{7}\epsilon^2 t$	$(-\frac{1}{2}, \frac{15}{14}, -\frac{2}{7}, -\frac{2}{7}, \frac{15}{14}, -\frac{1}{2}, -\frac{2}{7}, -\frac{2}{7})$	$p_1$
2/2	$\frac{20}{51}jt + \frac{23}{153}\epsilon jt + \frac{62}{153}\epsilon^2 t + \frac{8}{153}\epsilon^3 t$	$(-\frac{4}{9}, \frac{16}{17}, -\frac{1}{3}, -\frac{25}{153}, 1, -\frac{77}{153}, -\frac{1}{3}, -\frac{25}{153})$	$p_2$
2/1	$\frac{8}{153}jt + \frac{63}{153}\epsilon jt + \frac{23}{153}\epsilon^2 t + \frac{20}{51}\epsilon^3 t$	$(1, -\frac{25}{153}, -\frac{1}{3}, \frac{16}{17}, -\frac{4}{9}, -\frac{25}{153}, -\frac{1}{3}, -\frac{77}{153})$	$p_3$
4/1	$\frac{1}{20}jt + \frac{11}{40}\epsilon jt + \frac{1}{10}\epsilon^4 t + \frac{23}{40}\epsilon^5 t$	$(-\frac{11}{20}, \frac{1}{2}, -\frac{1}{5}, -\frac{3}{2}, \frac{6}{5}, \frac{1}{2}, -\frac{1}{5}, \frac{1}{4})$	$p_4$
4/0	$\frac{23}{40}jt + \frac{1}{10}\epsilon jt + \frac{1}{40}\epsilon^4 t + \frac{1}{20}\epsilon^5 t$	$(-\frac{1}{5}, \frac{1}{2}, -\frac{11}{20}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{2}, \frac{6}{5}, -\frac{3}{2})$	$p_5$
5/0	$\frac{1}{3}jt + \frac{1}{6}\epsilon jt + \frac{1}{6}\epsilon^5 t + \frac{1}{3}\epsilon^6 t$	$(\frac{1}{3}, \frac{1}{3}, -\frac{5}{3}, 1, \frac{1}{3}, \frac{1}{3}, 1, -\frac{5}{3})$	$p_6$
7/3	$\frac{1}{14}jt + \frac{3}{7}\epsilon jt + \frac{3}{7}\epsilon^7 t + \frac{1}{14}t$	$(\frac{1}{7}, -\frac{25}{14}, \frac{3}{2}, \frac{1}{7}, \frac{1}{7}, \frac{3}{2}, -\frac{25}{14}, \frac{1}{7})$	$p_7$

The formula for the obstruction cocycle (1.9) implies that there are signs  $s_1, \dots, s_7 \in \{1, -1\}$  such that

$$(2.1) \quad \mathbf{o}_{\mathbb{Q}_{32}}(f)(e) = \sum_{i=1}^7 s_i \|p_i\|.$$

The point classes in (2.1) do not depend on the embedding of the associated simplices because each intersection point  $p_i$  is contained in just one element of the arrangement. In this situation, in contrast to the Lovasz conjecture, we can not just apply Proposition 1.6.

To prove that the obstruction element does or does not vanish, the cohomology class of the obstruction cocycle inside  $H_2(W_n \setminus \bigcup \mathcal{A}; \mathbb{Z})_{\mathbb{Q}_{4n}}$  has to be computed. However, to prove that the obstruction element does not vanish, we are not compelled to completely identify the obstruction element.

With a help of the Poincaré-Alexander duality isomorphism and the Universal coefficient isomorphism we have

$$(2.2) \quad \begin{aligned} H_2(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z}) &\cong H^{n-4}(\bigcup \hat{\mathcal{A}}, \mathbb{Z}) \\ &\cong \text{Hom}\left(H_{n-4}\left(\bigcup \hat{\mathcal{A}}, \mathbb{Z}\right); \mathbb{Z}\right) \oplus \text{Ext}\left(H_{n-5}\left(\bigcup \hat{\mathcal{A}}\right), \mathbb{Z}\right) \end{aligned}$$

where  $\hat{\mathcal{A}}$  denotes the one-point compactification of the arrangement  $\mathcal{A}$ . The calculations of  $H_{n-4}(\bigcup \hat{\mathcal{A}}, \mathbb{Z})$  and  $\text{Ext}(H_{n-5}(\bigcup \hat{\mathcal{A}}), \mathbb{Z})$  can be carried out by the Ziegler-Živaljević formula [23]. For example, there is a decomposition

$$(2.3) \quad \begin{aligned} H_{n-4}(\bigcup \hat{\mathcal{A}}, \mathbb{Z}) &\cong \bigoplus_{V \in P} H_{n-4}(\Delta(P_{<V}) * S^{\dim V}, \mathbb{Z}) \\ &\cong \bigoplus_{d=0}^{n-4} \left( \bigoplus_{V \in P: \dim V = d} \tilde{H}_{n-5-d}(\Delta(P_{<V}), \mathbb{Z}) \right) \end{aligned}$$

where  $P$  is the intersection poset of the arrangement  $\mathcal{A}$ . By convention,  $\tilde{H}_{-1}(\emptyset) = \mathbb{Z}$ . The decomposition (2.3) is not a decomposition of  $\mathbb{Q}_{4n}$ -modules. This fact is illustrated in [4, Theorem 4.7.(C)]. If considered with the appropriate field coefficients (2.3) is a decomposition of  $\mathbb{Q}_{4n}$ -modules [11, Proposition 2.3]. In order to use (2.2) and (2.3) and to compute the coinvariants  $H_2(W_n \setminus \bigcup \mathcal{A}; \mathbb{Z})_{\mathbb{Q}_{4n}}$ , we have to keep in mind that Poincaré-Alexander duality isomorphism is not an isomorphism of  $\mathbb{Q}_{4n}$ -modules. Fortunately, it is a  $\mathbb{Q}_{4n}$ -map up to an orientation character. On the other hand, the universal coefficient isomorphism, when the Ext part vanishes, is a  $\mathbb{Q}_{4n}$ -map, if the action on Hom is given by

$$(g \cdot f)(x) = f(g^{-1} \cdot x)$$

for  $g \in \mathbb{Q}_{4n}$ ,  $f \in \text{Hom}\left(H_{n-4}\left(\bigcup \hat{\mathcal{A}}, \mathbb{Z}\right); \mathbb{Z}\right)$  and  $x \in H_{n-4}(\bigcup \hat{\mathcal{A}}, \mathbb{Z})$ .

Let a modified  $\mathbb{Q}_{4n}$  action  $*$  on  $\text{Hom}\left(H_{n-4}\left(\bigcup \hat{\mathcal{A}}, \mathbb{Z}\right); \mathbb{Z}\right)$  be defined by

$$(2.4) \quad (g * f)(x) = \det(g)f(g^{-1} \cdot x)$$

for  $g \in \mathbb{Q}_{4n}$ ,  $f \in \text{Hom}\left(H_{n-4}\left(\bigcup \hat{\mathcal{A}}, \mathbb{Z}\right); \mathbb{Z}\right)$  and  $x \in H_{n-4}\left(\bigcup \hat{\mathcal{A}}, \mathbb{Z}\right)$ . If the Ext part vanishes in (2.2) and the action on the Hom part is assumed to be  $*$ , the Poincaré duality isomorphism

$$H_2(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z}) \cong \text{Hom}\left(H_{n-4}\left(\bigcup \hat{\mathcal{A}}, \mathbb{Z}\right); \mathbb{Z}\right)$$

becomes an isomorphism of  $\mathbb{Q}_{4n}$ -modules. Consequently,

$$H_2(W_n \setminus \bigcup \mathcal{A}; \mathbb{Z})_{\mathbb{Q}_{4n}} \cong \text{Hom}\left(H_{n-4}\left(\bigcup \hat{\mathcal{A}}, \mathbb{Z}\right); \mathbb{Z}\right)_{\mathbb{Q}_{4n}}.$$

Now we identify every point class from the sum (2.1) using the isomorphism (2.2)

$$\varphi : H_2\left(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z}\right) \rightarrow \text{Hom}\left(H_{n-4}\left(\bigcup \hat{\mathcal{A}}, \mathbb{Z}\right); \mathbb{Z}\right).$$

The isomorphism is an evaluation of the linking number (when it is correctly defined). For point classes the following formula holds

$$\varphi(\|f(x)\|)(l) = \text{link}(l, \|f(x)\|)l$$

where  $l \in H_{n-4}(\bigcup \hat{\mathcal{A}}; \mathbb{Z})$ . Finally we must determine whether the class of

$$\varphi(\mathfrak{o}_{\mathbb{Q}_{4n}}(f)(e)) \in \text{Hom}\left(H_{n-4}\left(\bigcup \hat{\mathcal{A}}, \mathbb{Z}\right); \mathbb{Z}\right)$$

in the group of coinvariants  $\text{Hom}\left(H_{n-4}\left(\bigcup \hat{\mathcal{A}}, \mathbb{Z}\right); \mathbb{Z}\right)_{\mathbb{Q}_{4n}}$  is or is not zero. In each case this is a different, and usually a very difficult problem.

In the present situation, the exact sequence of  $\mathbb{Q}_{4n}$ -modules

$$0 \rightarrow \bigoplus_{V \in P: \dim V = n-4} \tilde{H}_{n-5-d}(\Delta(P_{<V}); \mathbb{Z}) \rightarrow H_{n-4}\left(\bigcup \hat{\mathcal{A}}, \mathbb{Z}\right)$$

induces an exact sequence of  $\mathbb{Q}_{4n}$ -modules

$$(2.5) \quad H_2\left(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z}\right) \xrightarrow{\chi} \text{Hom}\left(\bigoplus_{V \in P: \dim V = n-4} \tilde{H}_{n-5-d}(\Delta(P_{<V}); \mathbb{Z}), \mathbb{Z}\right) \rightarrow 0.$$

The geometric interpretation of the map  $\chi$  is the computation of the linking numbers. Let us assume that all of the  $(n-5-d)$  homology groups in the above sum are free. The left exactness of the coinvariant functor implies that the sequence

$$(2.6) \quad H_2\left(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z}\right)_{\mathbb{Q}_{4n}} \xrightarrow{\chi^*} \text{Hom}\left(\bigoplus_{V \in P: \dim V = n-4} \tilde{H}_{n-5-d}(\Delta(P_{<V}); \mathbb{Z}), \mathbb{Z}\right)_{\mathbb{Q}_{4n}} \rightarrow 0$$

is exact. The geometric interpretation of the map  $\chi^*$  is the summation of linking. The map  $\chi^*$  is a good test map for detecting whether an element  $\mathfrak{o} \in H_2(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z})_{\mathbb{Q}_{4n}}$  is not zero.

We use the map  $\chi^*$  (2.6) and prove that  $\chi^*([\mathfrak{o}_{\mathbb{Q}_{32}}(f)]) \neq 0$ . Let us identify groups in question (2.5) and (2.6).

LEMMA 2.5. (A)  $\bigoplus_{V \in P: \dim V = 4} \tilde{H}_{-1}(\Delta(P_{<V}); \mathbb{Z}) \cong \mathbb{Z}^4$ , as abelian groups.

(B)  $\text{Hom}\left(\bigoplus_{V \in P: \dim V = 4} \tilde{H}_{-1}(\Delta(P_{<V}); \mathbb{Z}), \mathbb{Z}\right)_{\mathbb{Q}_{32}} \cong \mathbb{Z}_2$ .

PROOF. Statement (A) follows from the Hasse diagram of the arrangement and the Ziegler-Živaljević formula. Statement (B) is a consequence of the set equality  $\epsilon^4 L = L$  and the following orientation computation: the element  $\epsilon^4$  acts on  $W_8$  by preserving its orientation. On the orthogonal complement  $L^\perp$  of  $L$  the operator  $\epsilon^4$ , for the basis  $\{e_1 + e_2 + e_3 + e_4, e_2 + e_3 + e_4 + e_5, e_3 + e_4 + e_5 + e_6, e_1 + \dots + e_8\}$  of  $L^\perp$ , has the matrix

$$\Xi = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since  $\det \Xi = -1$ , the element  $\epsilon^4$  changes the orientation of  $L^\perp$  and consequently it changes the orientation on  $L$ . If  $l \in \bigoplus_{V \in P: \dim V=4} \tilde{H}_{-1}(\Delta(P_{<V}); \mathbb{Z})$  is the generator associated with the subspace  $L$ , then the sum

$\bigoplus_{V \in P: \dim V=4} \tilde{H}_{-1}(\Delta(P_{<V}); \mathbb{Z})$  is generated as a free abelian group by elements  $l, \epsilon l, \epsilon^2 l, \epsilon^3 l$ . The set equality  $\epsilon^4 L = L$  implies the homology equality  $\epsilon^4 l = -l$ .

Let  $\xi \in \text{Hom} \left( \bigoplus_{V \in P: \dim V=4} \tilde{H}_{-1}(\Delta(P_{<V}); \mathbb{Z}), \mathbb{Z} \right)$  be given by  $\xi(l) = 1$  and  $\xi(\epsilon^i l) = 0$  for  $i \in \{1, 2, 3\}$ .

Then by (2.4)

$$\epsilon^4 * \xi(l) = \det(\epsilon^4) \xi(\epsilon^{-4} l) = \xi(\epsilon^4 l) = \xi(-l) = -1$$

and

$$\epsilon^4 * \xi(\epsilon^i l) = \det(\epsilon^4) \xi(\epsilon^{-4+i} l) = 0$$

for  $i \in \{1, 2, 3\}$ . Thus, there is a relation

$$\xi + \epsilon^4 * \xi = 0$$

which in coinvariants becomes  $2[\xi] = 0$ . The statement (B) follows.  $\square$

The proof of case (A) of Theorem 2.2 is a consequence of the following lemma.

LEMMA 2.6. (A) The element  $\chi(\mathfrak{o}_{\mathbb{Q}_{32}}(f))$  is given by

$$\begin{aligned} \chi(\mathfrak{o}_{\mathbb{Q}_{32}}(f))(l) &= s_5 + s_6, & \chi(\mathfrak{o}_{\mathbb{Q}_{32}}(f))(\epsilon^1 l) &= s_3 + s_4, \\ \chi(\mathfrak{o}_{\mathbb{Q}_{32}}(f))(\epsilon^2 l) &= s_1 + s_2, & \chi(\mathfrak{o}_{\mathbb{Q}_{32}}(f))(\epsilon^3 l) &= 1. \end{aligned}$$

$$(B) \chi^*([\mathfrak{o}_{\mathbb{Q}_{32}}(f)]) \in \text{Hom} \left( \bigoplus_{V \in P: \dim V=4} \tilde{H}_{-1}(\Delta(P_{<V}); \mathbb{Z}), \mathbb{Z} \right)_{\mathbb{Q}_{32}} \text{ is the generator of the group } \mathbb{Z}_2.$$

PROOF. Statement (A) follows from formula (2.1) and the associated table of intersections. The second statement is a consequence of the geometric interpretation of the map  $\chi^*$  - summation of the intersection numbers.  $\square$

Thus the obstruction element  $[\mathfrak{o}_{\mathbb{Q}_{32}}(f)]$  is not zero, and we have proved case (A) of Theorem 2.2.

2.2.2. Case  $n = 10$  and  $\alpha = (1, 2, 2, 1, 2, 2)$ . Since the proof of (B) follows the steps of the previous

case, we will just outline the computational parts which differ.

Let  $f : S^3 \rightarrow W_{10}$  be given by  $f(t) = (-\frac{21809669044}{3234846615}, \frac{23}{29}, \frac{19}{23}, \frac{17}{19}, \frac{13}{17}, \frac{11}{13}, \frac{7}{11}, \frac{5}{7}, \frac{3}{5}, \frac{2}{3})$ .

The arrangement  $\mathcal{A}$  is now a minimal  $\mathbb{Q}_{40}$ -arrangement containing the subspace  $L$  defined by

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= x_2 + x_3 + x_4 + x_5 + x_6 = x_3 + x_4 + x_5 + x_6 + x_7 \\ &= \sum_{i=1}^8 x_i = 0. \end{aligned}$$

The arrangement  $\mathcal{A}$  has five maximal elements  $L, \epsilon L, \epsilon^2 L, \epsilon^3 L$  and  $\epsilon^4 L$ . This follows from the set equality  $L = \epsilon^5 L$ . Let us determine the intersection of the image under  $f$  of the maximal cell

$$e = ([t, \epsilon t] \cup [\epsilon t, \epsilon^2 t] \cup [\epsilon^2 t, \epsilon^3 t] \cup \dots \cup [\epsilon^8 t, \epsilon^9 t]) * [jt, \epsilon jt]$$

with the test space  $\bigcup \mathcal{A} = L \cup \epsilon L \cup \epsilon^2 L \cup \epsilon^3 L \cup \epsilon^4 L$ . The results of  $10 \times 5 = 50$  intersections  $f([\epsilon^i t, \epsilon^{i+1} t] * [jt, \epsilon jt]) \cap \epsilon^r L$  can be summarized in the following way:

$$\begin{aligned} f([\epsilon^3 t, \epsilon^4 t] * [jt, \epsilon jt]) \cap \epsilon^2 L &= \{q_1\}, & f([\epsilon^5 t, \epsilon^6 t] * [jt, \epsilon jt]) \cap \epsilon^3 L &= \{q_2\}, \\ f([\epsilon^5 t, \epsilon^6 t] * [jt, \epsilon jt]) \cap \epsilon^4 L &= \{q_3\} \end{aligned}$$

and consequently  $\text{card}(f(e) \cap \bigcup \mathcal{A}) = 3$ . There are  $s_1, s_2, s_3 \in \{1, -1\}$  such that

$$(2.7) \quad \mathfrak{o}_{\mathbb{Q}_{40}}(f)(e) = s_1 \|q_1\| + s_2 \|q_2\| + s_3 \|q_3\|.$$

As in the previous case we use the map  $\chi^*$  (2.6) and prove that  $\chi^*([\mathfrak{o}_{\mathbb{Q}_{40}}(f)]) \neq 0$ . The set equality  $L = \epsilon^5 L$  implies that

$$(2.8) \quad \text{Hom} \left( \bigoplus_{V \in P: \dim V=6} \tilde{H}_{-1}(\Delta(P_{<V}); \mathbb{Z}), \mathbb{Z} \right)_{\mathbb{Q}_{40}} \cong \mathbb{Z}_2.$$

The element  $\epsilon^5$  acts on  $W_{10}$  by changing its orientation. On the orthogonal complement  $L^\perp$  of  $L$  the operator  $\epsilon^5$ , for the basis  $\{e_1 + \dots + e_5, e_2 + \dots + e_6, e_3 + \dots + e_7, e_4 + \dots + e_8, e_5 + \dots + e_9, e_6 + \dots + e_{10}\}$  of  $L^\perp$ , has the matrix

$$\Xi = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since  $\det \Xi = -1$ , the element  $\epsilon^5$  changes the orientation of  $L^\perp$  and consequently does not change the orientation on  $L$ . Let  $l \in H_6(\bigcup \hat{\mathcal{A}}; \mathbb{Z})$  be the generator associated to the subspace  $L$ , then the sum  $\bigoplus_{V \in P: \dim V=6} \tilde{H}_{-1}(\Delta(P_{<V}); \mathbb{Z})$  is generated as a free abelian group by elements  $l, \epsilon l, \epsilon^2 l, \epsilon^3 l, \epsilon^4 l$ . The set

equality  $L = \epsilon^5 L$  implies the homology equality  $\epsilon^5 l = l$ . Let  $\xi \in \text{Hom} \left( \bigoplus_{V \in P: \dim V=4} \tilde{H}_{-1}(\Delta(P_{<V}); \mathbb{Z}), \mathbb{Z} \right)$

be defined by  $\xi(l) = 1$  and  $\xi(\epsilon^i l) = 0$  for  $i \in \{1, 2, 3, 4\}$ . Then

$$(\epsilon^5 * \xi)(l) = \det(\epsilon^5) \xi(\epsilon^{-5} l) = -\xi(\epsilon^5 l) = -\xi(l) = -1$$

and

$$(\epsilon^5 * \xi)(\epsilon^i l) = \det(\epsilon^5) \xi(\epsilon^{-5+i} l) = -\xi(\epsilon^{-5+i} l) = 0$$

for  $i \in \{1, 2, 3, 4\}$ . Thus, there is a relation

$$\xi + \epsilon^5 * \xi = 0$$

which implies the isomorphism (2.8) we indicated. As before  $\chi^*([\mathfrak{o}_{\mathbb{Q}_{40}}(f)])$  is a generator of the group  $\mathbb{Z}_2$ . Therefore case (B) of Theorem 2.2 is proved.

We state the following conjecture.

**CONJECTURE 2.7.** All symmetric six-tuples  $(a, b, c, a, b, c)$  are solutions of problem 2.1.

### 3. The $(a, b, a)$ class of 3-fan 2-measures partitions

The problem discussed in this chapter will be managed by an extension of obstruction theory methods used in the paper [5]. Our result, although obtained along the lines of the same method, differs from the result of the paper [5] at the most critical phase of the proof. The search for the target extension space and the identification of the obstruction element is only remotely similar.

The appendix contains the ancillary material referred to in this section.

Let  $\mu_1, \mu_2, \dots, \mu_m$  be proper Borel probability measures on  $S^2$ . Let  $(\alpha_1, \alpha_2, \dots, \alpha_k) \in (\mathbb{R}_{>0})^k$  be a vector where  $\alpha_1 + \alpha_2 + \dots + \alpha_k = 1$ . The general problem stated in [2] is:

**PROBLEM 3.1.** Determine all triples  $(m, k, \alpha) \in \mathbb{N} \times \mathbb{N} \times \mathbb{R}^k$  such that for any collection of  $m$  measures  $\{\mu_1, \mu_2, \dots, \mu_m\}$ , there exists a  $k$ -fan  $(x; l_1, \dots, l_k)$  with the property

$$(\forall i = 1, \dots, k) (\forall j = 1, \dots, m) \mu_j(\sigma_i) = \alpha_i.$$

Such a  $k$ -fan  $(x; l_1, \dots, l_k)$  is called an  $\alpha$ -partition for the collection of measures  $\{\mu_1, \mu_2, \dots, \mu_m\}$ .

In this chapter we prove the existence of a class of solutions of the above problem in the case  $m = 2 / k = 3$ .

**THEOREM 3.2.** Let us choose  $\alpha = (a, b, a) \in \mathbb{R}_{>0}^3$  such that  $2a + b = 1$ . Then any two proper Borel probability measures  $\mu$  and  $\nu$  on the sphere  $S^2$  admit an  $\alpha$ -partition by a 3-fan  $\mathfrak{p} = (x; l_1, l_2, l_3)$ .

**3.1. The configuration space / test map scheme.** The CS / TM setting is the same as in the previous chapter except for the test space. Therefore we give only an outline. For details one can also consult [2], [5] and [4].

**The configuration space.** Let  $\mu$  and  $\nu$  be two proper Borel probability measures on  $S^2$ . The configuration space  $X_\mu$  associated with the measure  $\mu$  is

$$X_\mu = \{(x; l_1, \dots, l_n) \in F_n \mid (\forall i = 1, \dots, n) \mu(\sigma_i) = \frac{1}{n}\}.$$

As before,  $X_\mu$  is a Stiefel manifold  $V_2(\mathbb{R}^3)$  of all orthonormal 2-frames in  $\mathbb{R}^3$ .

**Test maps.** Let  $\alpha$ -vectors have the form  $\alpha = (\frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n}) \in \frac{1}{n} \mathbb{N}^3$ , where  $a_1 + a_2 + a_3 = n$ . The test map for this fan problem is defined by

$$\begin{aligned} \Phi_\nu : X_\mu &\rightarrow W_n = \{x \in \mathbb{R}^n \mid x_1 + \dots + x_n = 0\} \\ \Phi_\nu(\mathfrak{p}) &= (\nu(\sigma_1) - \frac{1}{n}, \dots, \nu(\sigma_n) - \frac{1}{n}). \end{aligned}$$

**The action.** The dihedral group  $\mathbb{D}_{2n} = \langle j, \varepsilon \mid \varepsilon^n = j^2 = 1, \varepsilon j = j \varepsilon^{n-1} \rangle$  acts both on the possible solution space  $X_\mu$  and the linear subspace  $W_n \subseteq \mathbb{R}^n$ , as in the previous chapter. Observe that  $X_\mu$  is  $\mathbb{D}_{2n}$ -homeomorphic to the manifold  $V_2(\mathbb{R}^3)$ , where  $V_2(\mathbb{R}^3)$  is a  $\mathbb{D}_{2n}$ -space given by

$$\varepsilon(x, y) = (x, R_x(\frac{2\pi}{n})(y)), \quad j(x, y) = (-x, y),$$

and  $R_x(\theta) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is the rotation around the axes determined by  $x$  through the angle  $\theta$ .

**The test space.** Given  $\alpha = (\frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n}) \in \frac{1}{n}\mathbb{N}^3$ , let  $L(\alpha) \subset W_n$  be defined by the equations

$$(3.1) \quad x_1 + \dots + x_{a_1} = x_{a_1+1} + \dots + x_{a_1+a_2} = x_{a_1+a_2+1} + \dots + x_n = 0.$$

The test space for this problem is the union  $\bigcup \mathcal{A}(\alpha) \subset W_n$  of the smallest  $\mathbb{D}_{2n}$ -invariant linear subspace arrangement  $\mathcal{A}(\alpha)$ , containing  $L(\alpha)$ .

Since the test map  $\Phi_\nu$  is  $\mathbb{D}_{2n}$ -equivariant, the following proposition holds.

**PROPOSITION 3.3.** Let  $\alpha = (\frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n}) \in \frac{1}{n}\mathbb{N}^3$  be a vector such that  $a_1 + a_2 + a_3 = n$ . If there is no  $\mathbb{D}_{2n}$ -equivariant map

$$\Phi : V_2(\mathbb{R}^3) \rightarrow W_n \setminus \bigcup \mathcal{A}(\alpha)$$

then for any two measures  $\mu$  and  $\nu$  on  $S^2$ , there exists an  $\alpha$ -partition  $(x; l_1, l_2, l_3)$  of measures  $\mu$  and  $\nu$ .

As we have seen in Proposition 2.4 and in [5] and [4] the Stiefel manifold  $V_2(\mathbb{R}^3)$  can be substituted with the sphere  $S^3$ . Thus we prove that there is no  $\mathbb{Q}_{4n}$ -map

$$S^3 \rightarrow W_n \setminus \bigcup \mathcal{A}(\alpha).$$

**3.2. Proof of Theorem 3.2.** Theorem 3.2 will be proved by showing that there is no  $\mathbb{Q}_{4n}$ -map  $S^3 \rightarrow W_n \setminus \bigcup \mathcal{A}(a, b, a)$ . The arrangement  $\mathcal{A}(a, b, a)$  is a minimal  $\mathbb{D}_{2n}$ -invariant arrangement containing the subspace  $L \subset W_n$  given by

$$x_1 + \dots + x_a = x_{a+1} + \dots + x_{a+b} = x_{a+b+1} + \dots + x_n = 0.$$

The codimension of the arrangement  $\mathcal{A}(a, b, a)$  inside  $W_n$  is two, and so the complement is connected. The fundamental group of the complement is far from trivial, so the obstruction theory can not be applied directly. We use the idea of a target extension scheme introduced in [5]. Let us denote by  $M$  the complement  $W_n \setminus \bigcup \mathcal{A}(a, b, a)$ .

**3.2.1. The target extension scheme.** The basic idea of the target extension scheme is to find a  $\mathbb{Q}_{4n}$ -space  $N$  which contains the target space  $M$ , and to prove that there is no  $\mathbb{Q}_{4n}$ -map  $S^3 \rightarrow N$ . This would imply that there is no  $\mathbb{Q}_{4n}$ -map  $S^3 \rightarrow M$ . In our case the target space  $M$  is the complement of the arrangement, and so we can refine the idea as follows:

- *Increase the codimension.* Let  $H$  be an arbitrary hyperplane in  $W_n$  and let  $\mathcal{B}$  be the minimal  $\mathbb{Q}_{4n}$ -invariant arrangement containing the subspace  $L \cap H$ . The inclusion  $\bigcup \mathcal{B} \subseteq \bigcup \mathcal{A}(a, b, a)$  implies that

$$(3.2) \quad W_n \setminus \bigcup \mathcal{B} \supseteq W_n \setminus \bigcup \mathcal{A}(a, b, a).$$

The dimension of maximal elements of the arrangement  $\mathcal{B}$  is  $n - 4$ . Let us denote by  $N$  the "new" complement  $W_n \setminus \bigcup \mathcal{B}$ .

- *Apply the general position map scheme.* The codimension of the arrangement  $\mathcal{B}$  inside  $W_n$  is three and so the complement  $N = W_n \setminus \bigcup \mathcal{B}$  is 1-connected. Therefore the question of the existence of a  $\mathbb{Q}_{4n}$ -map  $S^3 \rightarrow N$  depends only on the primary obstruction. We can now use the general position map scheme.

Unfortunately, the target extension scheme only provides hope that we can apply, once more, the mechanisms we already developed. The main problem is just shifted to the question of *how to find a hyperplane*  $H$  in a way that there is no  $\mathbb{Q}_{4n}$ -map  $S^3 \rightarrow N$ .

**3.2.2. In the pursuit of the hyperplane  $H$ .** There are two rough heuristics we can use. First, if we define a  $\mathbb{Q}_{4n}$ -map  $f : S^3 \rightarrow W_n$ , we can introduce  $H$  in such a way that

- (A)  $f$  becomes a map in general position for the new arrangement  $\mathcal{B}$ , and
- (B) the cardinality of the set  $f(S^3) \cap \bigcup \mathcal{B}$  is as small as can be achieved.

The second requirement for introduction of the hyperplane is that the group of coinvariants  $H_2(W_n \setminus \bigcup \mathcal{B}; \mathbb{Z})_{\mathbb{Q}_{4n}}$  has torsion. Nevertheless, the choice of the hyperplane  $H$  is strongly connected with the properties of the action of the group on the arrangement  $\mathcal{A}(a, b, a)$ . In this situation we are going to exploit the following symmetry  $jL = L$ . Observe that element  $j \in \mathbb{Q}_{4n}$  changes the orientation of the orthogonal complement  $L^\perp$ .

**3.2.3. The map in general position.** Let us define a  $\mathbb{Q}_{4n}$ -map  $f : S^3 \rightarrow W_n$  and introduce  $H$  so that  $f$  becomes a map in general position. Let  $S^3$  be the simplicial complex  $P_{2n} * P_{2n}$ , where  $P_{2n}$  is the regular  $2n$ -gon. Let  $t$  be a fixed vertex in one of the copies of  $P_{2n}$ . Let  $u_i = e_i - \frac{1}{n} \sum_{j=1}^n e_j$ ,  $i \in \{1, \dots, n\}$ , where

$e_1, \dots, e_n$  are elements of the standard basis of  $\mathbb{R}^n$ . We define  $f : S^3 \rightarrow W_n$  on the vertex  $h(t) = u_1$  and then extend it equivariantly. This implies

$$h(\epsilon^i t) = \epsilon^i h(t) = u_{(i+1) \bmod n} \quad h(\epsilon^i j t) = \epsilon^i j h(t) = u_{i \bmod n}.$$

In the future all the indexes in  $W_n$  will be calculated mod  $n$ .

NOTATION 3.4. Every 3-simplex of the sphere  $P_{2n} * P_{2n}$ , determined by four vertices, has form

$$(3.3) \quad [\epsilon^p t, \epsilon^{p+1} t; \epsilon^q j t, \epsilon^{q+1} j t]$$

and will be denoted by  $\sigma_{p,q}$ .

The list of all simplices of the form  $[u_i, u_{i+1}; u_j, u_{j+1}]$  which intersect the subspace  $L$  is given in the following table

Simplex	Bar. coor. of the intersection	for $r \in$
1. $[u_a, u_{a+1}; u_{a+b}, u_{a+b+1}]$	$\frac{1}{n}(a, \beta, \gamma, a); \beta + \gamma = b$	
2. $[u_a, u_{a+1}; u_r, u_{r+1}]$	$\frac{1}{n}(a, b, \gamma, \delta); \gamma + \delta = a$	$[a + b + 1, n - 3]$
3. $[u_a, u_{a+1}; u_{n-2}, u_{n-1}]$	$\frac{1}{n}(a, b, \gamma, \delta); \gamma + \delta = a$	
4. $[u_a, u_{a+1}; u_{n-1}, u_n]$	$\frac{1}{n}(a, b, \gamma, \delta); \gamma + \delta = a$	
5. $[u_r, u_{r+1}; u_{a+b}, u_{a+b+1}]$	$\frac{1}{n}(\alpha, \beta, b, a); \alpha + \beta = a$	$[2, a - 2]$
6. $[u_1, u_2; u_{a+b}, u_{a+b+1}]$	$\frac{1}{n}(\alpha, \beta, b, a); \alpha + \beta = a$	
7. $[u_n, u_1; u_a, u_{a+1}]$	$\frac{1}{n}(\alpha, \beta, b, a); \alpha + \beta = a$	
8. $[u_n, u_1; u_{a+b}, u_{a+b+1}]$	$\frac{1}{n}(a, b, \gamma, \delta); \gamma + \delta = a$	
9. $[u_n, u_1; u_r, u_{r+1}]$	$\frac{1}{n}(a, \beta, \gamma, a); \beta + \gamma = b$	$[a + 1, a + b - 2]$
10. $[u_n, u_1; u_{a+b-1}, u_{a+b}]$	$\frac{1}{n}(a, \beta, \gamma, a); \beta + \gamma = b$	

If we introduce the hyperplane  $H$  by

$$H = \{\mathbf{x} \in \mathbb{R}^n \mid x_{a+b} - x_{a+1} + x_1 - x_{n-1} = 0\},$$

the list of simplices now intersecting  $L_1 = L \cap H$  reduces to

(A) two simplices, for  $a > b$ ,

$$\rho_1 = [u_a, u_{a+1}; u_{a+b}, u_{a+b+1}] \text{ and } \rho_2 = [u_n, u_1; u_a, u_{a+1}],$$

with barycentric coordinates of intersection points  $\frac{1}{n}(a, \frac{b}{2}, \frac{b}{2}, a)$  and  $\frac{1}{n}(b, a - b, b, a)$ ;

(B) one simplex, for  $b > a$ ,  $\rho_1 = [u_a, u_{a+1}; u_{a+b}, u_{a+b+1}]$ , with barycentric coordinates of intersection points  $\frac{1}{n}(a, \frac{b}{2}, \frac{b}{2}, a)$ .

In order to simplify the exposition let us assume that  $b > a$ . Case (A) is actually a consequence of Case (B). Therefore,

$$f(S^3) \cap L_1 = \{\mathbf{y} = \frac{a}{n}u_a + \frac{b}{2n}u_{a+1} + \frac{b}{2n}u_{a+b} + \frac{a}{n}u_{a+b+1}\}.$$

We have made extensive use of the first heuristic. It is not easy to see that we also used the second heuristic. Let us just say that the following property will be a significant ingredient in the interpretation of the obstruction element.

(3.4) Element  $j$  does NOT change the orientation of the subspace  $(L_1 \cap jL_1)^\perp$ .

3.2.4. *The obstruction cocycle.* There are 8 simplices in  $P_{2n} * P_{2n}$  which are in the inverse image under  $f$  of the simplex  $[u_a, u_{a+1}; u_{a+b}, u_{a+b+1}]$ . These simplices are

$$\begin{aligned} \sigma_{a-1, a+b}; \quad \sigma_{n+a-1, a+b}; \quad \sigma_{a-1, n+a+b}; \quad \sigma_{n+a-1, n+a+b}; \\ \sigma_{a+b-1, a}; \quad \sigma_{n+a+b-1, a}; \quad \sigma_{a+b-1, n+a}; \quad \sigma_{n+a+b-1, n+a}. \end{aligned}$$

All eight simplices are in the orbit of the following two simplices from the maximal cell  $e$  (consult appendix 4.2):

$$\Theta_1 = \sigma_{2a-1, 0} \text{ and } \Theta_2 = \sigma_{b-1, 0}.$$

The points  $\xi_1 \in \Theta_1$  and  $\xi_2 \in \Theta_2$ , with barycentric coordinates  $(a, \frac{b}{2}, \frac{b}{2}, a)$  and  $(\frac{b}{2}, a, a, \frac{b}{2})$ , are mapped inside the arrangement  $\mathcal{B}$ , precisely

$$f(\xi_1) \in \epsilon^a(L_1 \cap jL_1) \text{ and } f(\xi_2) \in \epsilon^{-a}(L_1 \cap jL_1).$$

Therefore, formula (1.9) implies that the obstruction cocycle is

$$\begin{aligned} \circ_{\mathbb{Q}_{4n}}(f)(e) &= \|f(\xi_1)\| + \|f(\xi_2)\| = \det(\epsilon^a)\epsilon^a \|\mathbf{y}\| + \det(\epsilon^{-a})\epsilon^{-a} \|\mathbf{y}\| \\ &= \det(\epsilon^{-a}) (\det(\epsilon^{2a})\epsilon^a \|\mathbf{y}\| + \epsilon^{-a} \|\mathbf{y}\|) \end{aligned}$$

Since we are interested in the cohomology / coinvariant class of the obstruction cocycle (1.8), instead of  $\mathfrak{o}_{\mathbb{Q}_{4n}}(f)(e)$  we can look at the element

$$(3.5) \quad \mathfrak{o}'_{\mathbb{Q}_{4n}} = 2 \det(\epsilon^{-a}) \|\mathbf{y}\|.$$

REMARK 3.5. In contrast to the previous chapter, the point class  $\|\mathbf{y}\|$  **depends** on the placement of the simplex  $\rho_1 = [u_a, u_{a+1}; u_{a+b}, u_{a+b+1}]$  with respect to the arrangement  $\mathcal{B}$ . In terms of paper [5],  $\|\mathbf{y}\|$  is the broken point class.

3.2.5. *The obstruction element.* It remains to prove that  $2\|\mathbf{y}\|$  does not vanish when we pass to coinvariants. The brief proof of this fact would be: There is a symmetry

$$j[u_a, u_{a+1}; u_{a+b}, u_{a+b+1}] = [u_a, u_{a+1}; u_{a+b}, u_{a+b+1}]$$

which does not change the orientation of the simplex and therefore gives the relation  $j\|\mathbf{y}\| = \|\mathbf{y}\|$  and NOT  $\|\mathbf{y}\| = -j\|\mathbf{y}\|$ . This is the only relation which could provide in coinvariants that  $2(\text{class } \|\mathbf{y}\|) = 0$ . Thus  $2\|\mathbf{y}\|$  does not vanish in coinvariants and the obstruction element is NOT zero. We proved Theorem 3.2 for all rational triples  $(a, b, a)$ .

Just to be safe, let us analyze the point class  $\|\mathbf{y}\|$ , its placement with the respect to the arrangement  $\mathcal{B}$  and the indicated  $j$  symmetry. Observe that we are not forced to determine the complete group of coinvariants  $H_2(W_n \setminus \bigcup \mathcal{B}; \mathbb{Z})_{\mathbb{Q}_{4n}}$ . We only have to prove that  $2(\text{class } \|\mathbf{y}\|) \neq 0$ .

Translating the simplex  $[u_a, u_{a+1}; u_{a+b}, u_{a+b+1}]$  by a small generic vector and then intersecting it with the arrangement, we find that the boundary (of the shaded region) links with the elements of the arrangement as in the Figure 3.

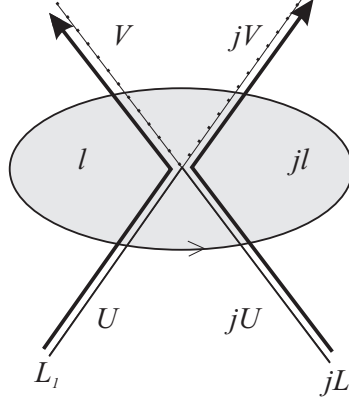


FIGURE 3. The position of a point class relative to the arrangement

Figure 3 indicates two set relations  $L_1 = U \cup jV$  and  $jL_1 = jU \cup V$ . Since element  $j$  does not change the orientation of the subspace  $(L_1 \cap jL_1)^\perp$ , the geometric generators of the group  $H_{n-4}(\bigcup \hat{\mathcal{B}}; \mathbb{Z})$  (the dual of  $H_2(W_n \setminus \bigcup \mathcal{B}; \mathbb{Z})$ ) can only be given by sets

$$(3.6) \quad L_1, \epsilon L_1, \dots, \epsilon^{n-1} L_1, jL_1, \epsilon jL_1, \dots, \epsilon^{n-1} jL_1 \text{ and } U \cup V, \epsilon(U \cup V), \dots, \epsilon^{n-1}(U \cup V).$$

This fact is a consequence of the following observations

- the orientation on the arrangement should be consistent with the group action,
- the boundaries of the two pieces which are glued together have opposite orientations,
- the element  $j(U \cup V)$  is not a generator because it is a linear combination of already introduced generators.

The isomorphism composition of the Poincaré duality isomorphism and the Universal coefficient isomorphism for the arrangement  $\mathcal{B}$ ,

$$\varphi : H_2(W_n \setminus \bigcup \mathcal{B}) \rightarrow \text{Hom}\left(H_{n-4}\left(\bigcup \hat{\mathcal{B}}, \mathbb{Z}\right); \mathbb{Z}\right),$$

is the computation of the linking number with the elements of the basis (3.6). Thus,

$$\begin{aligned} \varphi(\|\mathbf{y}\|)(l) &= \pm 1, & \varphi(\|\mathbf{y}\|)(jl) &= \mp 1, & \varphi(\|\mathbf{y}\|)(t) &= 1, \\ \varphi(\|\mathbf{y}\|)(u) &= 0, & & & \text{for all other basis elements,} \end{aligned}$$

where  $l, t \in H_{n-5}(\bigcup \widehat{\mathcal{B}}, \mathbb{Z})$  are elements of a basis determined by sets  $L_1$  and  $U \cup V$ . Since  $\varphi(\|\mathbf{y}\|)(t) = 1$ , the class  $2[\varphi(\|\mathbf{y}\|)]$  does not vanish. The obstruction element is NOT zero and we have proved Theorem 3.2 for all rational triples  $(a, b, a)$ .

**3.2.6. The closing argument.** Let  $S \subseteq \mathbb{R}_{>0}^3$  be the space of all triples  $(a, b, c)$ ,  $a + b + c = 1$  such that for  $\alpha = (a, b, c)$  there exists an  $\alpha$ -partition of measures  $\mu$  and  $\nu$  by a 3-fan. Since we assumed that our measures  $\mu$  and  $\nu$  are two proper Borel probability measures, the space  $S$  is a closed subset of  $\mathbb{R}_{>0}^3$ . In this notation, so far we have proved the inclusion

$$\{(\frac{a}{n}, \frac{b}{n}, \frac{a}{n}) \in \mathbb{Q}_{>0}^3 \mid 2a + b = n, a, b \in \mathbb{Z}\} \subseteq S.$$

Therefore

$$\{(a, b, c) \in \mathbb{R}_{>0}^3 \mid a + b + c = 1\} = \text{cl}\{(\frac{a}{n}, \frac{a+b}{n}, \frac{b}{n}) \mid 2a + 2b = n, a, b \in \mathbb{Z}\} \subseteq S.$$

where  $\text{cl}$  denotes the closure of the subset in  $\mathbb{R}^3$ . The theorem is proved.

#### 4. Appendix: Geometry of $\mathbb{Q}_{4n}$

**4.1. Generalized quaternion group.** The sphere  $S^3 = S(\mathbb{H}) = Sp(1)$  can be seen as the group of all unit quaternions. Consider a root of unity  $\epsilon = \epsilon_{2n} = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n} \in S(\mathbb{H})$ . The group generated by  $\epsilon$  is a subgroup of  $S(\mathbb{H})$  of order  $2n$ . The generalized quaternion group, [8, p. 253], is a subgroup of order  $4n$  generated by  $\epsilon$  and  $j$ . The group  $\mathbb{Q}_{4n}$  acts on  $S^3$  as a subgroup, and on  $W_n$  via the already defined  $\mathbb{D}_{2n}$ -action by the quotient homomorphism  $\mathbb{Q}_{4n} \rightarrow \mathbb{Q}_{4n}/\{1, -1\} \cong \mathbb{D}_{2n}$ . The  $\mathbb{Q}_{4n}$ -action on  $S^3$  is **free**. Let  $H = \{1, \epsilon^n\} = \{1, -1\} \subset \mathbb{Q}_{4n}$ . Then the quotient group  $\mathbb{Q}_{4n}/H$  is isomorphic to the dihedral group  $\mathbb{D}_{2n}$  of order  $2n$ . Also, the group  $H$  acts on  $W_n$  and on  $\mathbb{R}^n$  trivially. Since the group  $\mathbb{Q}_{4n}$  acts on  $S^3$  as a subgroup and  $S^3$  is a connected group, then the  $\mathbb{Q}_{4n}$ -module structure  $\mathcal{Z}$  on  $H_3(S^3, \mathbb{Z})$  must be trivial.

**4.2. The natural  $\mathbb{Q}_{4n}$  cellular / simplicial structure on  $S^3$ .** Let the circle  $S^1$  be represented by the simplicial complex of the regular  $2n$ -gon  $P_{2n}$ . Then the sphere  $S^3$ , as a simplicial complex, is the join  $P_{2n}^{(1)} * P_{2n}^{(2)}$  of two copies of  $P_{2n}$ . Denote the vertices of  $P_{2n}^{(1)}$  and  $P_{2n}^{(2)}$  by  $a_1, \dots, a_{2n}$  and  $b_1, \dots, b_{2n}$ , respectively. The action of the group  $\mathbb{Q}_{4n}$  on  $S^3$  is defined on vertices by

$$\epsilon \cdot a_i = a_{i \bmod 2n+1}, \epsilon \cdot b_i = b_{i \bmod 2n+1}, j \cdot a_1 = b_1$$

and it extends equivariantly to the upper skeletons. The associated chain complex  $\mathfrak{C} = \{C_i\}$  has the form

$$0 \longrightarrow \mathbb{Z}^{4n^2} \longrightarrow \mathbb{Z}^{8n^2} \longrightarrow \mathbb{Z}^{4n^2+4n} \longrightarrow \mathbb{Z}^{4n} \longrightarrow 0.$$

**4.3. An economic  $\mathbb{Q}_{4n}$  cellular structure on  $S^3$ .** It comes from the minimal resolution of  $\mathbb{Z}$  by free  $\mathbb{Q}_{4n}$ -modules described in [8, p. 253]. The associated cellular complex has one  $\mathbb{Q}_{4n}$  0-cell  $a$ , two  $\mathbb{Q}_{4n}$  1-cells  $b$  and  $b'$ , two  $\mathbb{Q}_{4n}$  2-cells  $c$  and  $c'$ , and then finally one  $\mathbb{Q}_{4n}$  3-cell  $e$ . The associated chain complex  $\mathfrak{D} = \{D_i\}$  has the form

$$0 \rightarrow \mathbb{Z}[\mathbb{Q}_{4n}]e \xrightarrow{\partial} \mathbb{Z}[\mathbb{Q}_{4n}]c \oplus \mathbb{Z}[\mathbb{Q}_{4n}]c' \xrightarrow{\partial} \mathbb{Z}[\mathbb{Q}_{4n}]b \oplus \mathbb{Z}[\mathbb{Q}_{4n}]b' \xrightarrow{\partial} \mathbb{Z}[\mathbb{Q}_{4n}]a \rightarrow 0$$

where

$$\begin{aligned} \partial e &= (\epsilon - 1)c - (\epsilon j - 1)c' & \partial c &= (1 + \dots + \epsilon^{n-1})b - (j + 1)b' & \partial b &= (\epsilon - 1)a \\ & & \partial c' &= (\epsilon j + 1)b + (\epsilon - 1)b' & \partial b' &= (j - 1)a \end{aligned}$$

Thus, it will be enough to look at the obstruction cocycle  $c_{\mathbb{Q}_{4n}}(h)$  on the maximal cell  $e$ , and to prove that its image is or is not zero, when we pass to the cohomology.

**4.4. The chain map  $\mathfrak{D} \rightarrow \mathfrak{C}$ .** There exists a cellular map  $f : \mathfrak{D} \rightarrow \mathfrak{C}$ . Since we are interested in 3-cochains we only need to know the concrete expression for the cellular map on the top dimensional cell  $e$ ,

$$f(e) = [a_1, \epsilon a_1] * [b_1, \epsilon b_1] + [\epsilon a_1, \epsilon^2 a_1] * [b_1, \epsilon b_1] + \dots + [\epsilon^{n-1} a_1, \epsilon^n a_1] * [b_1, \epsilon b_1],$$

where the simplices on the right hand side are appropriately oriented.

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